Perturbative coefficients for improved actions by Monte Carlo at large β

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Perturbative estimates of operator coefficients for improved lattice actions are becoming increasingly important for precision simulations of many hadronic observables. Following previous work by Dimm, Lepage, and Mackenzie, we consider the feasibility of computing operator coefficients from numerical simulations deep in the perturbative region of lattice theories. Here we introduce a background field technique that may allow for the computation of the coefficients of clover-field operators in a variety of theories. This method is tested by calculations of the renormalized quark mass in lattice NRQCD, and of the $O(\alpha_s)$ clover coefficient for Sheikholeslami-Wohlert fermions. First results for the coefficient of the magnetic moment operator in NRQCD are also presented.

Recent simulations of many hadronic observables using tadpole-improved actions, on both coarse and fine lattices, have yielded promising results. This has spurred further development of ever more highly improved lattice actions. At the same time, it has also become apparent that one must go beyond tree-level improvement in order to obtain precision results for many quantities.

The increasing diversity and complexity of improved actions makes traditional perturbative calculations of operator coefficients in these actions problematic. A simpler alternative is needed, if only to provide reasonable estimates of these coefficients. Dimm, Lepage, and Mackenzie have shown that the renormalized mass for Wilson fermions, and the energy zero in NRQCD, can be calculated from numerical simulations, deep in the perturbative region of these theories [1].

In this work we introduce a background field technique for perturbative matching of lattice theories by numerical simulation. We use this technique to estimate the coefficients of clover-field operators for both Sheikholeslami-Wohlert [2] and NRQCD [3] fermions.

The clover coefficients are tuned by computing quark propagators in a uniform background magnetic field, and matching the spin-flip energy

$$\Delta E(gB) \equiv E(\downarrow, gB) - E(\uparrow, gB) \qquad (1)$$

in the lattice theory with the same quantity in continuum QCD. The continuum calculation has been done by Sapirstein [4], using the background field formalism [5]. We used a lattice formulation of the background field method. This method has the important feature that the combination gB of bare coupling g and bare field strength B does not get renormalized. Thus lattice and continuum results for ΔE can be directly compared, without regard to the different regulators that are used.

We used the Wilson gluon action, and split the link variable into a classical background part and a fluctuating dynamical part, $U_{\mathrm{tot},\mu}(x) = U_{\mathrm{dyn},\mu}(x) U_{\mathrm{cl},\mu}(x)$. For an Abelian background (as used in the continuum [4]), $U_{\mathrm{cl},\mu=2}(x_1) = e^{igBx_1\lambda_3}$. Periodic boundary conditions also require $U_{\mathrm{cl},\mu=1}(x_1 = N, x_2) = e^{-igBNx_2\lambda_3}$, and $gB = 2\pi n_{\mathrm{ext}}/N^2$, where n_{ext} is an integer.

This background field is an exact solution of the equations of motion for the effective action, with *zero* external current. This is because the only covariant vector operators that can be formed from the classical field strength $F_{\mathrm{cl},\mu\nu}$ are all identically zero (e.g., $D_\mu[A_{\mathrm{cl}}]F_{\mathrm{cl},\mu\nu} = 0$). Consequently we do not need to make a perturbative calculation of the external current required to support a more general classical field.

On the other hand, a given background field does not minimize the effective action. In a numerical simulation one observes "tunneling" of

the system, from the extremum at $F_{\rm cl}$, to the minimum at zero field. To compute observables in the presence of the classical field we modify the path integral, introducing a "current" J into the action as an intermediate step, in order to create a minimum in the effective potential near $F_{\rm cl}$:

$$Z[J] \equiv \int [dU_{\rm dyn}]\, e^{-\beta(S_{\rm Wil}[U_{\rm dyn}U_{\rm cl}]+\lambda_J J[U_{\rm dyn}])}. \quad (2)$$

Physical quantities in the presence of the classical field are obtained by extrapolation in the coupling, $\lambda_J \to 0$. However, one must perform this extrapolation from sufficiently large λ_J, in order for the system to be dominated by fluctuations in the dynamical fields near the classical field.

To test this procedure, the NRQCD spin flip energy was computed twice, using two currents, $J_1 = \sum_{x,\mu} \text{Re-Tr}[U_{{\rm dyn},\mu}(x)]$, and $J_2 = (1/gB)\sum_{x,\mu} \text{Im-Tr}[U_{{\rm dyn},\mu} F_{{\rm cl},12}]$. The extrapolated results agree, as shown in Fig. 1.

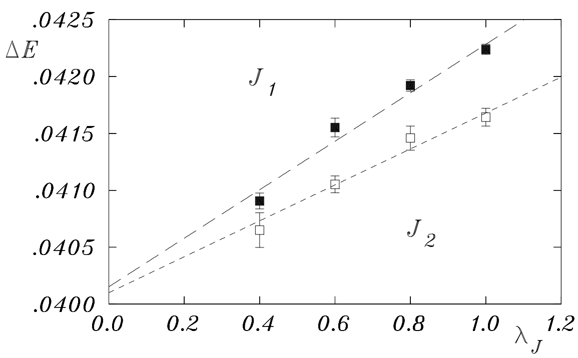

Figure 1. NRQCD spin-flip energy for two external currents ($\beta = 16$, $M_0 = 1.8$).

The action (and hence the current J) must be invariant under background field transformations,

$$U_{{\rm cl},\mu}(x) \to \Omega(x) U_{\rm cl} \Omega^\dagger(x+\hat\mu),$$
$$U_{{\rm dyn},\mu}(x) \to \Omega(x) U_{\rm dyn} \Omega^\dagger(x), \quad (3)$$

in order to preserve the nonrenormalization of gB. Likewise, we require a background field-covariant gauge fixing prescription for the dynamical fields, when computing quark propagators:

$$\langle 0|T\left(\psi(x)\bar\psi(0)\right)|0\rangle =$$
$$\frac{1}{Z(J)}\int [dU_{\rm dyn}]\, e^{-\beta(S_{\rm W}+\lambda_J J)} K^{-1}\big[U^G_{\rm dyn}U_{\rm cl};x\big]\ ,(4)$$

where $U^G_{\rm dyn}$ is the gauge-fixed dynamical field. We gauge fix with respect to transformations of the dynamical field (with fixed classical field)

$$U_{{\rm dyn},\mu}(x) \to \Omega(x) U_{\rm dyn} U_{\rm cl}(x) \Omega^\dagger(x+\hat\mu) U^\dagger_{\rm cl}(x). \ (5)$$

We used a background field axial ($U_{{\rm dyn},4} = I$) plus Coulomb ($\sum_{i=1}^3 D[U_{\rm cl}]_i U_{{\rm dyn},i} = 0$) gauge fixing [1]. These lattice transformations and gauge fixings have the desired continuum limits [5] (note that the quantum symmetry, Eq. (5), is broken by the external current just as in the continuum).

We used this approach to calculate the renormalized quark mass $M_{\rm ren}$ in lattice NRQCD. Relativistic corrections up to $O(v^4)$ in the mean quark velocity [3] were included. The renormalized masses were extracted from quark correlation functions with small nonzero three-momentum. Sample results are shown in Fig. 2. All simulations were done on $12^3 \times 24$ lattices at $\beta = 9$, with $n_{\rm ext} = 2$, and current J_1 was used. Simulations were done both with and without tadpole improvement (using the average plaquette to compute u_0 in the former case).

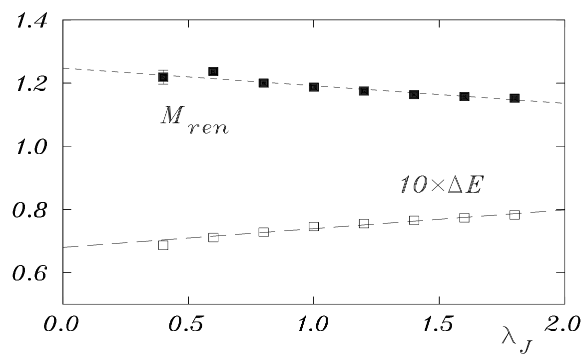

Figure 2. NRQCD renormalized quark mass and spin-flip energy vs. λ_J ($\beta = 9$, $M_0 = 1.18$, with tadpole improvement).

We did linear extrapolations to $\lambda_J = 0$ to obtain the "true" masses, given in Tables 1 and 2. Errors include a systematic error due to the extrapolation. The Monte Carlo ("MC") estimates of the masses are in good agreement with results from perturbation theory ("PT"), due to Morningstar [6]. Notice that most of the renormaliza-

tion is due to tadpoles, which is clearly resolved by the Monte Carlo results.

Table 1
Renormalized quark mass and clover coefficient in lattice NRQCD for three bare masses M_0, *without* tadpole improvement.

M_0	$M_{\rm ren}$ (MC)	$M_{\rm ren}$ (PT)	$c_4^{(1)}$ (MC)
2.65	2.90(3)	2.94	4.8(2)
1.90	2.16(3)	2.20	4.0(2)
1.18	1.52(3)	1.51	2.3(2)

Table 2
NRQCD renormalizations *with* tadpole improvement.

M_0	$M_{\rm ren}$ (MC)	$M_{\rm ren}$ (PT)	$c_4^{(1)}$ (MC)
2.65	2.69(3)	2.73	1.3(2)
1.90	1.95(3)	1.98	0.9(2)
1.18	1.24(3)	1.26	0.6(2)

Simulation results for ΔE in NRQCD are also shown in Fig. 2. We find the coefficient of the magnetic moment operator, c_4, by matching the (extrapolated) lattice ΔE to the continuum value [4]. Defining $c_4 = 1 + c_4^{(1)}\alpha_s$, and using $\alpha_s = \alpha_V(\pi/a)$, we obtain the results in Tables 1 and 2.

To calibrate our NRQCD calculations, we analyzed spin-flip energies for Sheikholeslami-Wohlert (SW) fermions, where the clover coefficient c_{sw} is known [7]. We used a tree-level clover coefficient, and the value determined non-perturbatively for massless fermions in Ref. [7] ($c_{sw} = 1.24$ at $\beta = 9$). We worked at modest quark masses ($ma \approx 0.5$) at which $O((ma)^2)$ errors may dominate over $O(\alpha_s ma)$ errors. This prevents a direct matching of the lattice results with the continuum. Instead, we compare the simulation results for ΔE with the spin-flip energy for "classical" SW fermions [$U_{\rm dyn} = I$, $c_{sw} = 1$, and with the classical mass identified with $E(\uparrow, gB, \lambda_J)$ measured in the simulation]. We expect the effects of $O(a^2)$ errors will roughly cancel in this approximate matching procedure.

Sample results are shown in Fig. 3. The simulation results with a tree-level clover coefficient show a significant departure from the matching condition. At this quark mass ($ma \approx .5$), a value of c_{sw} somewhat smaller than the coefficient for massless fermions [7] is suggested by these results.

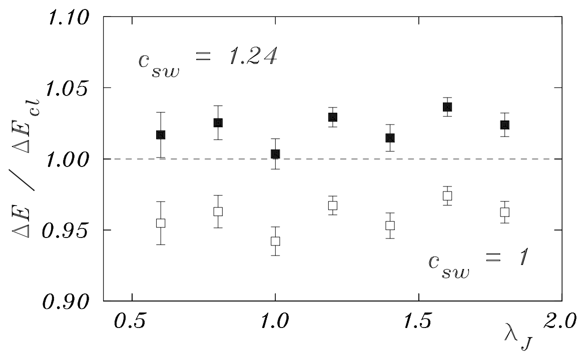

Figure 3. Monte Carlo ΔE for SW fermions, compared to the "classical" spin-flip energy $\Delta E_{\rm cl}$ ($\beta = 9$, $\kappa = .115$).

In summary, we have introduced a background field method for simulations in the perturbative region of lattice theories. Encouraging preliminary results were obtained for the coefficients of clover operators for SW and NRQCD fermions, and for the renormalized quark mass in NRQCD.

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